

ON MUCKENHOUT-WHEEDEN CONJECTURE

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ABSTRACT. Let M denote the dyadic Maximal Function. We show that there is a weight w , and Haar multiplier T for which the following weak-type inequality fails.

$$\sup_{t>0} tw(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq C \int_{\mathbb{R}} |f| Mw(x) dx.$$

(With T replaced by M , this is a well-known fact.) This shows that a dyadic version of the so-called Muckenhoupt-Wheeden Conjecture is false. This is accomplished by using current techniques in weighted inequalities to show that a particular L^2 consequence of the inequality above does not hold.

1. INTRODUCTION

The starting point of this work goes back to 1971 [7], when C. Fefferman and E. Stein, in order to establish vector-valued estimates for the maximal function, proved that if w is a weight, namely a non-negative locally integrable function, and M denotes the Hardy-Littlewood maximal operator then

$$\sup_{t>0} tw(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq c \int_{\mathbb{R}^n} |f| Mw(x).$$

A very natural question was then raised by B. Muckenhoupt and R. Wheeden (see [15]): could we replace the Hardy-Littlewood maximal operator M by a Calderón-Zygmund operator? Their conjecture, known as Muckenhoupt-Wheeden Conjecture, is stated below.

Conjecture 1.1. (*Muckenhoupt-Wheeden*) *Let w be a weight and M be the Hardy-Littlewood maximal operator. Let T be a Calderón-Zygmund operator with $\|T\|_{CZO} \leq 1$. Then*

$$(1.2) \quad \sup_{t>0} tw(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq C \int_{\mathbb{R}} |f| Mw(x) dx$$

The exact definition of Calderón-Zygmund operator need not concern us here, though it certainly includes the non-positive Hilbert transform (see chapter VII of [18] for precise definitions). The hope was that the Conjecture identified a somewhat robust principle. We herein disprove the *dyadic version* of this conjecture. So, M is replaced by the (smaller) dyadic maximal function, and T will be a Haar multiplier, which are the simplest possible dyadic Calderón-Zygmund operators.

Endpoint estimates are known to be the most delicate ones, and very frequently they are also the most powerful. That is the case of Muckenhoupt-Wheeden Conjecture. For instance,

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an extrapolation result due to D. Cruz-Uribe and C. Pérez [5] shows this: If w is a weight and (1.2) holds with T a sublinear operator then

$$(1.3) \quad \int_{\mathbb{R}} |T(f)|^p w(x) dx \leq \int_{\mathbb{R}} |f|^p \left(\frac{Mw}{w} \right)^p w(x) dx.$$

The dyadic version of this result is also true (see [5], Remark 1.5).

With a few partial results that we shall discuss later in the introduction, the Muckenhoupt-Wheeden Conjecture has been open up to today's date. In this paper, we answer the dyadic version of (1.2) in the negative by disproving (1.3). We are ready to state our main theorem.

Theorem 1.4. *There exist a weight w and a Haar multiplier T for which T is unbounded as map from $L^2\left(\left(\frac{Mw}{w}\right)^2 w\right)$ to $L^2(w)$.*

As a corollary we solve a long-standing conjecture,

Corollary 1.5. *Muckenhoupt-Wheeden conjecture in its dyadic version is false.*

For the proof we construct a measure w and a Haar multiplier T that avoids all cancellations. The tool behind this construction is the corona decomposition, that has proven to be very useful in finding sharp estimates when the weight is in the A_p class [9–12].

Throughout the literature, there has been evidence for a positive answer to the conjecture as well as for a negative one. S. Chanillo and R. Wheeden [3] showed that a square function satisfied the Muckenhoupt-Wheeden Conjecture. We also mention the work of Buckley [1], who in dimension n proved that (1.2) holds for weights $w_\delta(x) = |x|^{-n(1-\delta)}$ for $0 < \delta < 1$.

The sharpest results in this direction are due to C. Pérez [16]: If T is a Calderón-Zygmund operator and M^2 is the Hardy-Littlewood maximal operator iterated 2 times,

$$\sup_{t>0} tw(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq C \int_{\mathbb{R}} |f| M^2 w(x) dx.$$

He actually proved something better, M^2 can be replaced by the smaller operator $M_{L(\log L)^\epsilon}$. In an attempt to understand these endpoint estimates, A. Lerner, S. Ombrosy and C. Pérez considered a somehow "dual" problem of Muckenhoupt-Wheeden, we refer the reader to [13]. A negative answer to (1.2) was provided by M.J. Carro, C. Pérez, F. Soria and J. Soria when T is a fractional integral, [2]. There are two points that distinguish this example from the singular integral one: 1) the lack of cancellation when treating positive operators and 2) the construction depends upon T being a true fractional integral and does not allow an immediate extension to the singular integral case.

By imposing an extra condition on the weight w , a weaker version of Muckenhoupt-Wheeden can be formulated. This is known as the Weak Muckenhoupt-Wheeden Conjecture and appears in work of A. Lerner, S. Ombrosy and C. Pérez [15], [14].

Conjecture 1.6. *(Weak Muckenhoupt-Wheeden) Let w be an A_1 weight and let $\|w\|_{A_1}$ be the A_1 constant associated to it. Let T be a Calderón-Zygmund operator with $\|T\|_{CZO} \leq 1$.*

Then

$$(1.7) \quad \sup_{t>0} tw(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq C\|w\|_{A_1} \int_{\mathbb{R}} |f|w(x)dx$$

Recall that w is an A_1 weight if there exists a constant $c > 0$ such that $Mw(x) \leq cw(x)$ a.e. The smallest of such constants c is denoted by $\|w\|_{A_1}$. Thus, the Weak Muckenhoupt-Wheeden Conjecture would be an immediate consequence of (1.2), were it true. The continuity of Calderón-Zygmund operators in $L^1 \mapsto L^{1,\infty}$ when w is an A_1 weight is well known and goes back to the origins of the weighted theory with R. Hunt, B. Muckenhoupt and R. Wheeden [8] in dimension 1 and R. Coifman and C. Fefferman in higher dimensions [4]. The novelty of (1.7) resides with the linear dependence on $\|w\|_{A_1}$. Linear growth of the A_1 constant has been proven in the strong case for $p > 1$, that is

$$\|T\|_{L^p(w) \mapsto L^p(w)} \leq C(p, n)\|w\|_{A_1},$$

and this is sharp. The result was first proven by R. Fefferman and J. Pipher for $p \geq 2$ and T a classical singular integral operator in [6]. Later on it was extended to $p > 1$ and general Calderón-Zygmund operators by A. Lerner, S. Ombrosy and C. Pérez in [15]. The proof of A. Lerner *et al.* provides not only linear dependence on the A_1 constant, but also explicit dependence of the operator norm on p . The explicit dependence on p allows one to get the weak endpoint below,

$$\sup_{t>0} tw(\{x \in \mathbb{R} : |Tf(x)| > t\}) \leq C\|w\|_{A_1}(1 + \log\|w\|_{A_1}) \int_{\mathbb{R}} |f|w(x)dx.$$

We want to point out that even though this estimate is far from proving (1.7), it is the best result known up to this date. The Weak Muckenhoupt-Wheeden Conjecture remains open and we will not make any new contribution to it in this paper, but we are hoping to shed some light in the understanding of these endpoint estimates.

The paper is organized as follows. In Section 2 we introduce the necessary concepts. Section 3 presents an important tool in the proof of the result, the corona decomposition associated to a measure w . We dedicate Section 4 to the inductive construction of measures w and Haar multipliers T . Section 5 includes the proof of the main theorem. Finally, the last Section contains the bibliography.

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2. BASIC CONCEPTS

The space we will be working on is \mathbb{R} . Throughout the paper $|\cdot|$ will stand for the Lebesgue measure in \mathbb{R} , 1_E will be the characteristic function associated to the set $E \subset \mathbb{R}$, and for $x \geq 0$, $[x]$ denotes the integer part. The letters i, j, l, k will stand for positive integers. C will denote a universal constant, not necessarily the same in each case.

In the sequel when referring to M we will understand the dyadic maximal function, i.e., for $f \in L^1_{loc}$

$$Mf(x) = \sup_{Q \text{ dyadic}} \frac{1_Q}{|Q|} \int_Q f(x) dx.$$

We will use a different formulation of the two weight inequality (1.3). This characterization was first introduced by E. Sawyer in [17] and have been used since then, becoming one of the standard approaches. The proof is a well known exercise that we are not including in this paper.

Proposition 2.1. *Let w, v be two positive Borel measures, continuous with respect to Lebesgue measure and let T be a sublinear operator. Let C be a universal constant and $1 < p < \infty$, the statements below are equivalent,*

$$(2.2) \quad \begin{aligned} & \|Tf\|_{L^p(w)} \leq C \|f\|_{L^p(v)}, \\ & \|T(f\sigma)\|_{L^p(w)} \leq C \|f\|_{L^p(\sigma)}, \quad \sigma = v^{1-p'} 1_{\text{supp}(v)}. \end{aligned}$$

Remark 2.3. This new formulation provides a more symmetric estimate for T^* , the dual operator with respect to Lebesgue measure, i.e., (2.2) is equivalent to

$$(2.4) \quad \|T^*(fw)\|_{L^{p'}(\sigma)} \leq C \|f\|_{L^{p'}(w)},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

The estimate we want to disprove is a particular case of a 2-weighted inequality. We will work with weights $w \geq 0$, $w \in L^1_{loc}$ and $\sigma = \frac{w}{(Mw)^2}$, which is the dual measure of $v = \left(\frac{Mw}{w}\right)^2 w$. Throughout the paper σ will take the above form. The operators we consider are discrete dyadic operators. Let us recall some of the basic concepts associated to them before getting to the precise definition.

Definition 2.5. Let \mathcal{D} be the usual dyadic grid in \mathbf{R} , namely

$\mathcal{D} = \{ [2^k m, 2^k(m+1)) , \ m, k \in \mathbf{Z} \}$. Let $I = [a, b]$ be an interval in \mathcal{D} , then $I^- = [a, \frac{a+b}{2})$ is the *left child* of I and $I^+ = [\frac{a+b}{2}, b)$ is the *right child* of I . We define the L^2 -normalized Haar function associated to I , h_I as

$$h_I = \frac{1_{I^+} - 1_{I^-}}{|I|^{1/2}}$$

Our interest will lay on particular examples of dyadic operators, the Haar multipliers.

Definition 2.6. Let $\epsilon = (\epsilon_I)_{I \in \mathcal{D}}$ be a bounded sequence. The operator T_ϵ is a *Haar multiplier* associated to ϵ iff

$$T_\epsilon f = \sum_{I \in \mathcal{D}} \epsilon_I \langle f, h_I \rangle h_I.$$

3. THE CORONA DECOMPOSITION

This Section provides the tool to, given w , decompose measure σ . At the same time, the corona decomposition allows to group the dyadic intervals into families and consequently decompose any dyadic operator into a sum of operators. This was previously done in [12].

Definition 3.1. Let $\mathcal{D}' \subset \mathcal{D}$ be any collection of dyadic cubes. Call $(\mathcal{L} : \mathcal{D}'(L))$ a *corona decomposition of \mathcal{D}' relative to measure w* if these conditions are met. Let $L, L' \in \mathcal{L}$, we have

$$\frac{w(L')}{|L'|} \geq 4 \frac{w(L)}{|L|}, \quad L' \subsetneq L.$$

Define $\Gamma : \mathcal{D}' \rightarrow \mathcal{L}$ by requiring that $\Gamma(I)$ be the minimal element of \mathcal{L} that contains I . We set $\mathcal{D}'(L) := \{I \in \mathcal{D}' : \Gamma(I) = L\}$. Then for all $I \in \mathcal{D}'(L)$ we require

$$4 \frac{w(L)}{|L|} > \frac{w(I)}{|I|}.$$

Remark 3.2. The corona decomposition is obtained by a stopping time argument. It is for this reason that we will refer to \mathcal{L} as the *stopping collection* in the corona decomposition.

The collections $\mathcal{D}'(L)$ partition \mathcal{D}' . Since decompositions of dyadic intervals correspond directly to decompositions of dyadic operators, we can write any Haar multiplier as

$$(3.3) \quad T_\epsilon = \sum_{L \in \mathcal{L}} T_L \quad \text{where} \quad T_L = \sum_{I \in \mathcal{D}'(L)} \epsilon_I \langle f, h_I \rangle h_I.$$

We now focus on the structure of the measure σ . We will denote \mathcal{L}_0 to be the set of maximal intervals in \mathcal{L} . In general, we denote \mathcal{L}_j to be the maximal intervals on $\mathcal{L} \setminus \bigcup_{i=0}^{j-1} \mathcal{L}_i$.

Definition 3.4. Let $E \in \mathcal{D}(L)$, $L \in \mathcal{L}_j$, we define the *take away the children* operator on sets of \mathbb{R} as $\Delta_1 E = E \setminus \bigcup_{L' \in \mathcal{L}_{j+1}, L' \subset E} L'$. In general, we define

$$\Delta_l E = \bigcup_{\tilde{L} \in \mathcal{L}_{j+l-1}, \tilde{L} \subset E} \tilde{L} \setminus \bigcup_{L' \in \mathcal{L}_{j+l}, L' \subset E} L' \quad \text{for } l > 1.$$

Remark 3.5. This last definition helps us track the value of Mw . Notice that for every $x \in \Delta_l E$, $E \in \mathcal{D}(L)$, and $L \in \mathcal{L}$, we have $8^l \frac{w(L)}{|L|} \geq Mw(x) \geq 4^l \frac{w(L)}{|L|}$. Since $\{\Delta_l E\}_{l \geq 1}$ forms a partition of E , we can estimate

$$(3.6) \quad \sigma(E) = \int_E \frac{w}{Mw^2}(x) dx \geq \sum_{l=1}^{\infty} 8^{-2l} \left(\frac{|L|}{w(L)} \right)^2 w(\Delta_l E)$$

4. THE INDUCTIVE CONSTRUCTION

In this Section, we describe the inductive procedure that will provide measures w_k and operators T_k , the key elements in proving Theorem 1.4. We start with a few definitions associated to the base case.

Definition 4.1. Let $J = [a, a + \alpha)$ be a dyadic interval, we define the *jumping point* of J , and we denote it by $\text{jp}(J)$, as $\text{jp}(J) := a + \frac{\alpha}{3}$. We also denote the right end point of J by $\text{rep}(J) := a + \alpha$.

Notice that the “jumping point” divides the interval into two intervals of lengths one-third and two-thirds the length of the original interval, thus the “jumping point” has a periodic binary expansion, which fact is important to the construction. We now define the measure that gives name to the jumping point.

Definition 4.2. Let J be a dyadic interval and $\lambda > 0$ be a height, we define the measure associated to J with height λ as

$$\mu_J^\lambda = \lambda 1_{[\text{jp}(J), \text{rep}(J))}.$$

Having listed the key ingredients to construct our measure w_k inductively, we now focus on those associated to the construction of the operator T^k .

Definition 4.3. Let $k \geq 1$ be a fixed integer. Let J be a dyadic interval, we define Ξ_J as the following collection of intervals associated to J ,

$$(4.4) \quad \Xi_J := \{J = I_0 \supseteq \dots \supseteq I_{2k} : \text{jp}(J) \in I_i^-, |I_i| = 4|I_{i+1}|\}.$$

We denote $I(J) := I_{2k}$, the minimal interval in the collection Ξ_J . And we define the collection of the right children of the intervals in Ξ_J as

$$\Xi_J^+ := \{I_i^+ : I_i \in \Xi_J \setminus I(J)\}.$$

We are now ready to define the Haar multiplier associated to J with sign r_J , $S_{J,r(J)}$, as

$$(4.5) \quad S_{J,r(J)}(f) := r_J \sum_{I \in \Xi_J} \langle f, h_I \rangle h_I.$$

where $r_J \in \{+1, -1\}$.

Remark 4.6. There are two ideas about the jumping point that should be clarified. (1) If $\text{jp}(J) \in I$, then either $\text{jp}(J) \in I^-$ or $\text{jp}(J) \in I^+$. Moreover, these two events alternate, i.e., let $I \subset I' \subset J$ with $|I| = \frac{1}{2}|I'|$ and $\text{jp}(J) \in I, I'$. We have that if $\text{jp}(J) \in I'^-$ (respectively I'^+) then $\text{jp}(J) \in I^+$ (respectively I^-). This phenomenon explains that the chosen intervals in (4.4) satisfy $|I_i| = 4|I_{i+1}|$. (This is the consequence of the binary expansion of $1/3$.) (2) We take advantage of the localization of the jumping point in another way. The intervals $I \in \Xi_J^+$ “almost” form a partition of the support of μ_J^λ . If $k \rightarrow \infty$, they will actually form a partition, since we consider only a fixed number of them, we can only get

$$(4.7) \quad \text{supp} \mu_J^\lambda = [\text{jp}(J), \text{rep}(I(J))) \cup \bigcup_{I \in \Xi_J^+} I.$$

Remark 4.8. Notice that Ξ_J and consequently Ξ_J^+ and $S_{J,r(J)}$ depend on the parameter k , that will play the role of the induction index in the proof of Proposition 4.13. For the sake of simplicity, we omit the parameter k in the notation of those objects.

The following lemma takes advantage of the lack of cancellation in $S_{J,r(J)}(\mu_J^\lambda)$ to compare the μ_J^λ measure of a level set associated to $S_{J,r(J)}$ with that of J .

Lemma 4.9. *Let $k \geq 1$ be a fixed integer, J be a dyadic interval and μ_J^λ and $S_{J,r(J)}$ as above. Then, first the inner products $\langle \mu_J^\lambda, h_{I_i} \rangle$, for $I_i \in \Xi_J$ depend only on the numbers $\mu_J^\lambda(I(J))$ and $\{\mu_J^\lambda(I) : I \in \Xi_J^+\}$. Second,*

$$(4.10) \quad \mu_J^\lambda \left(\left\{ x : |S_{J,r(J)}\mu_J^\lambda(x)| > k \frac{\mu_J^\lambda(J)}{|J|} \right\} \right) \geq \frac{1}{4} 2^{-4k} \mu_J^\lambda(J).$$

Proof. For the first claim, $\langle \mu_J^\lambda, h_{I_i} \rangle$ depends only on the measure μ_J^λ assigned to the two children for I_i . And, these two children are unions of the sets in (4.7).

Turning to the second claim, let J be a dyadic interval, for any $I_i \in \Xi_J$ we have the following equality

$$(4.11) \quad \frac{\langle \mu_J^\lambda, h_{I_i} \rangle}{\sqrt{|I_i|}} = \frac{1/2\lambda|I_i| - (1/2 - 1/3)\lambda|I_i|}{|I_i|} = \frac{\lambda}{3}.$$

Since $I_{i+1} \subset I_i^-$ for all i , $\langle \mu_J^\lambda, h_{I_i} \rangle h_{I_i}$ is constant on I_{i+1} . Therefore, using (4.11) for every $x \in I(J)^-$

$$|S_{J,r(J)}\mu_J^\lambda(x)| = \sum_{i=0}^{2k} \frac{\langle \mu_J^\lambda, h_{I_i} \rangle}{\sqrt{|I_i|}} 1_{I(J)^-} = \frac{(2k+1)\lambda}{3}.$$

On the other hand,

$$\frac{\mu_J^\lambda(J)}{|J|} = \frac{\frac{2}{3}\lambda|J|}{|J|} = \frac{2}{3}\lambda,$$

and $\frac{(2k+1)\lambda}{3} > \frac{2}{3}k\lambda$ trivially. This added to the fact that $|S_{J,r(J)}\mu_J^\lambda(x)| \leq \frac{2}{3}k\lambda$ for all $x \in J$ and $x \notin I(L)^-$ gives,

$$(4.12) \quad I(L)^- = \left\{ x : |S_{J,r(J)}\mu_J^\lambda(x)| > k \frac{\mu_J^\lambda(J)}{|J|} \right\},$$

and

$$\mu_J^\lambda \left(\left\{ x : |S_{J,r(J)}\mu_J^\lambda(x)| > k \frac{\mu_J^\lambda(J)}{|J|} \right\} \right) \geq \mu_J^\lambda(I(J)^-) = \frac{1}{6}\lambda 2^{-4k}|J| = \frac{1}{4}2^{-4k}\mu_J^\lambda(J),$$

as desired. \square

Actually this estimate (4.10) is unimprovable, as follows from the John-Nirenberg inequality. (Our point of view in this construction is informed by the extension of the John-Nirenberg inequality in the weighted setting, as established in the work of the author with M. Lacey and S. Petermichl [12, page 137].)

We have shown that we can construct a particular Haar multiplier, that with respect to Lebesgue measure has no cancellation, and we have reversed the John-Nirenberg inequality. The success of this proof is based upon the observation that we can iterate this construction

on the elements of the partition in (4.7). Namely, we are free to change the measure μ_j^λ provided we *do not change the numbers* $\mu_j^\lambda(I)$, for $I \in \Xi_j^+$. And so, we will change the definition of μ_j^λ , without changing its total measure, in such a way that we carefully track the corona, so that we have (3.6). This means that at a different threshold, and a different part of our Haar multiplier, we will have a second reversal of the John-Nirenberg inequality. This construction will then have to be iterated many times, to overcome the exponential nature of the John-Nirenberg inequality. All of these considerations are incorporated into this proposition.

Proposition 4.13. *Let $k \geq 1$ be a fixed integer. There exist a family of random Haar multipliers T^k and a weight $w_k \not\equiv 0$, $w_k \in L_{loc}^1$ such that*

$$\sum_{L \in \mathcal{L}, L \subset [0,1]} w_k \left(\left\{ x : |T_L^k w_k(x)| > k \frac{w_k(L)}{|L|} \right\} \right) \geq c w_k([0,1]),$$

where \mathcal{L} is the stopping collection in the corona decomposition associated to measure w_k as defined in (3.1) and $c = 1/6$.

For the proof we need this definition.

Definition 4.14. Let $k \geq 1$ be an integer, J be a dyadic interval and Ξ_J be as above. We define the set of intervals $\mathcal{L}(J)$ as

$$\mathcal{L}(J) := \{L'(I) := I^{--} : I \in \Xi_J^+\}$$

Notice that the map

$$(4.15) \quad \begin{aligned} \Psi : \Xi_J^+ &\longmapsto \mathcal{L}(J) \\ I &\longmapsto L'(I) \end{aligned}$$

is a bijection and $|L'(I)| = \frac{1}{4}|I|$. Moreover, given J dyadic interval,

$$(4.16) \quad I(J) \cap L' = \emptyset \quad \text{for all } L' \in \mathcal{L}(J).$$

Remark 4.17. The passage to the ‘left-left child’ above helps us keep track of the corona. We will rescale all of the measure assigned to $I \in \Xi_J^+$ to ‘right two-thirds’ of I^{--} . Now, $\frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}$, so to preserve measure, we will need to multiply by 6. This explains the powers of 6 that appear below.

Proof. The proof follows from an inductive procedure. Let \mathcal{L}_0 , μ_0 and S^0 be as follows, $\mathcal{L}_0 = \{[0,1)\}$, $\mu_0 := \mu_{[0,1)}^1$ and $S^0 := S_{[0,1),r[0,1)}$. For a picture of the first stage see figure 1 below. In general, for every $j \geq 1$ we define

$$\begin{aligned} \mathcal{L}_j &= \bigcup_{L \in \mathcal{L}_{j-1}} \mathcal{L}(L) \\ \mu_j &= \sum_{i=0}^{j-1} \sum_{L \in \mathcal{L}_i} \mu_{I(L)}^{6^i} + \sum_{L' \in \mathcal{L}_j} \mu_{L'}^{6^j} \end{aligned}$$

$$S^j = S^{j-1} + \sum_{L' \in \mathcal{L}_j} S_{L', r(L')}.$$

For the proof of the proposition the selection of signs $r(L)$ is irrelevant. See figure 2 for a descriptive drawing of the second stage of the construction.

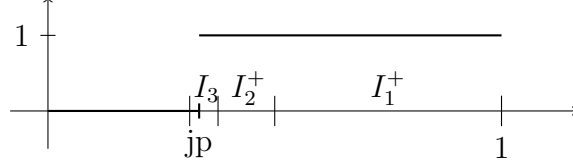


FIGURE 1. The first stage of the construction. The measure is held constant to the left of the jumping point, and on the interval I_3 . It recurses on the positive half of the other intervals: I_1^+, I_2^+ .

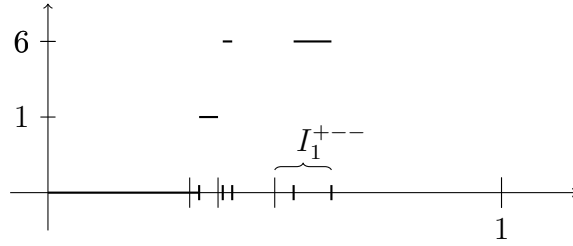


FIGURE 2. The second stage of the construction. Note that the horizontal scale is logarithmic. We repeat the first stage on the most left dyadic grandchild of each I_i^+ , we jump to height 6 so that the total measure of I_i^+ is preserved.

The next lemma states the main properties of the construction, namely, that the support of the measure built at each stage has shrunk with respect to the previous one. At the same time the new measure preserves the total measure and the measure of the intervals playing a role in previous stages. It is for this reason that we will refer to it as the ‘measure preserving lemma’. Its proof is postponed until Section 5.

Lemma 4.18. (*Measure Preserving Lemma*) *Let \mathcal{L}_j , μ_j and S^j as above for any $j \geq 1$. Let $L \in \mathcal{L}_j$, we have the following estimates*

- (1) $\sum_{L \in \mathcal{L}_j} |L| \leq \frac{(1-2^{-4k})^j}{6^j},$
- (2) $I(L) \cap L' = \emptyset$ for all $L' \in \mathcal{L}_i, L' \neq L, i \geq 1$. In particular $I(L) \cap I(L') = \emptyset,$
- (3) $\mu_{j+1}|_{[0,1] \setminus \cup_{L \in \mathcal{L}_j} L} = \mu_j|_{[0,1] \setminus \cup_{L \in \mathcal{L}_j} L},$
- (4) $\mu_{j+1}(I) = \mu_{j+1}(L'(I)) = \mu_j(I)$ for all $I \in \Xi_L^+,$
- (5) $\mu_i(I(L)) = \mu_j(I(L))$ for all $i \geq j,$
- (6) $\mu_i(L) = \mu_j(L)$ for all $i \geq j,$

$$(7) \quad \mu_i \left(\left\{ |S_{L,r(L)} \mu_i(x)| > k \frac{\mu_i(L)}{|L|} \right\} \right) = \mu_j \left(\left\{ |S_{L,r(L)} \mu_j(x)| > k \frac{\mu_j(L)}{|L|} \right\} \right) \quad \text{for all } i \geq j.$$

where in the last line $S_{L,r(L)}$ is as in (4.5).

We are now ready to prove the proposition. Let $M = M(k) > 0$ be an integer that will depend on k and will be chosen later. Let $T^k := S^M$ and $w_k := \mu_M$. We claim that the corona decomposition of \mathcal{D}' associated to μ_l is $(\mathcal{L} : \mathcal{D}'(L))$ where $\mathcal{L} = \bigcup_{i=0}^l \mathcal{L}_i$ and \mathcal{D}' is the set of dyadic intervals contained in $[0, 1)$. The first stopping interval in the corona decomposition associated to μ_0 is $[0, 1)$. It is easy to see that this is actually the only stopping interval, therefore \mathcal{L}_0 is the stopping collection in the corona decomposition associated to μ_0 and the claim is true in this case. The two facts that allow us to conclude the claim in general are: (a) parts (3) and (6) of Lemma 4.18, which let us keep the corona of the measure μ_j when we move to stage $j + 1$. Using backwards induction we have proved $\bigcup_{i=0}^{l-1} \mathcal{L}_i$ forms part of the stopping intervals associated to μ_l .

(b) For the next stage of the corona: due to parts (4) and (5) of Lemma 4.18 and (4.7), it is enough to consider the children of $I \in \Xi_L^+$ for all $L \in \mathcal{L}_{l-1}$. We have these relations:

$$\begin{aligned} \mu_l(I^+) &= 0 \\ \frac{\mu_l(I^-)}{|I^-|} &< 4 \frac{\mu_l(L)}{|L|} \\ \mu_l(I^{-+}) &= 0 \\ \frac{\mu_l(I^{--})}{|I^{--}|} &\geq 4 \frac{\mu_l(L)}{|L|} \\ \frac{\mu_l(J)}{|J|} &< 4 \frac{\mu_l(I^{--})}{|I^{--}|}, \quad \text{for all } J \subset I^{--} \end{aligned}$$

which describe the new level of stopping intervals as $\{L'(I) : I \in \Xi_L^+, L \in \mathcal{L}_{l-1}\}$. It is easy to check that this is the last stage. Herein we have proved that the corona decomposition associated to w_k is $(\mathcal{L} : \mathcal{D}'(L))$ with $\mathcal{L} = \bigcup_{i=0}^M \mathcal{L}_i$.

We now decompose the operator as described in (3.3), $T^k = \sum_{L \in \mathcal{L}} T_L^k$. Notice that $T_L^k = S_{L,r(L)}$ as defined in (4.5). We finish the proof using parts (4), (6) and (7) of Lemma 4.18 together with Lemma 4.9.

$$\begin{aligned} \sum_{\substack{L \in \mathcal{L} \\ L \subset [0,1]}} w_k \left(\left\{ x : |T_L^k w_k(x)| > k \frac{w_k(L)}{|L|} \right\} \right) &= \sum_{i=0}^M \sum_{L \in \mathcal{L}_i} w_k \left(\left\{ |S_{L,r(L)} w_k(x)| > k \frac{w_k(L)}{|L|} \right\} \right) \\ &= \sum_{i=0}^M \sum_{L \in \mathcal{L}_i} \mu_i \left(\left\{ |S_{L,r(L)} \mu_i(x)| > k \frac{\mu_i(L)}{|L|} \right\} \right) \\ &= \sum_{i=0}^M \sum_{L \in \mathcal{L}_i} \mu_L^{6^i} \left(\left\{ |S_{L,r(L)} \mu_L^{6^i}(x)| > k \frac{\mu_L^{6^i}(L)}{|L|} \right\} \right) \end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=0}^M \sum_{L \in \mathcal{L}_i} \frac{1}{6} 6^i 2^{-4k} |L| \\
&= \frac{1}{6} 2^{-4k} \sum_{i=0}^M (1 - 2^{-4k})^i \\
&= \frac{1}{6} (1 - (1 - 2^{-4k})^{M+1}) > \frac{1}{6} w_k([0, 1]),
\end{aligned}$$

for $M = \left\lceil \frac{\log(3)}{\log(1-2^{-4k})-1} \right\rceil + 1$. Note that M behaves exponentially with respect to k , which is exactly what one should expect from (4.10). □

5. PROOF OF MAIN THEOREM

Using Proposition 2.1 and (2.4), we can reduce the proof of the main theorem to finding a weight w , a Haar multiplier T and a function $f \in L^2(w)$ such that for all $C > 0$

$$\int_{\mathbb{R}} |T(fw)|^2 \sigma(x) dx \geq C \int_{\mathbb{R}} |f|^2 w(x) dx.$$

There is one more reduction, we can use a gliding hump argument to deduce the infinitary inequality above from the following finitary one,

Lemma 5.1. *(Main Lemma) For all $k \geq 1$, there exist a Haar multiplier T^k and a weight $w_k \not\equiv 0$, such that*

$$\int_{[0,1)} |T^k(w_k 1_{[0,1)})(x)|^2 \sigma_k(x) dx \geq C k^2 w_k([0, 1]),$$

where C is a universal constant.

Proof. Let $k \geq 1$ be a fixed natural number. Define T^k and w_k as in Proposition 4.13 and let $(\mathcal{L} : \mathcal{D}(L))$ be the corona decomposition associated to w_k . We now use the decomposition of T^k and σ_k as suggested in (3.3) and (3.6) respectively. It is now time to determine the more convenient choices of signs r_L that appear in the definition of (4.5). By Khintchine's inequalities we can find a sequence of signs $\{r_L\}_{L \in \mathcal{L}}$, $r_L \in \{+1, -1\}$ so that the first inequality below holds. That together with Chebyshev's inequality and Proposition 4.13 provide the desired estimate.

$$\begin{aligned}
\int_{[0,1)} |T^k(w_k 1_{[0,1)})(x)|^2 \sigma_k(x) dx &\geq \sum_{L \in \mathcal{L}} \int_{[0,1)} |T_L^k(w_k 1_{[0,1)})(x)|^2 \sigma_k(x) dx \\
&\geq \sum_{L \in \mathcal{L}} \left(\frac{w_k(L)}{|L|} \right)^2 k^2 \sigma_k \left(\left\{ T_L^k w_k > k \frac{w_k(L)}{|L|} \right\} \right) \\
&\geq \frac{k^2}{64} \sum_{L \in \mathcal{L}} w_k \left(\Delta_1 \left(\left\{ T_L^k w_k > k \frac{w_k(L)}{|L|} \right\} \right) \right)
\end{aligned}$$

$$\geq \frac{k^2}{64} \sum_{L \in \mathcal{L}} w_k \left(\left\{ T_L^k w_k > k \frac{w_k(L)}{|L|} \right\} \right) \geq C k^2 w_k([0, 1)),$$

where C is a universal constant. In the last line, we have used (4.12) and property (2) of Lemma 4.18. \square

6. PROOF OF THE MEASURE PRESERVING LEMMA

Proof. Let us start proving (1). A backwards induction argument allows us to reduce the problem to proving that given $L \in \mathcal{L}_{j-1}$ and for any $j \geq 1$,

$$(6.1) \quad \sum_{L' \subset L, L' \in \mathcal{L}_j} |L'| = \frac{1}{6}(1 - 2^{-4k})|L|.$$

The proof of (6.1) uses $|L'(I)| = \frac{1}{4}|I|$, with $L'(I)$ defined in (4.15).

$$\begin{aligned} \sum_{L' \subset L, L' \in \mathcal{L}_j} |L'| &= \frac{1}{4} |\text{rep}(I(L)), \text{rep}(L)| \\ &= \frac{1}{4} \left[\frac{2}{3}|L| - \frac{2}{3}|I(L)| \right] \\ &= \frac{1}{6}(1 - 2^{-4k})|L|. \end{aligned}$$

Proof of (2). Let $L, L' \in \mathcal{L}$, we need to distinguish two cases.

- (1) If $L \cap L' = \emptyset$ then $I(L) \cap L' = \emptyset$ trivially.
- (2) Suppose $L' \subset L$, then there exist i and j , $i > j$ such that $L' \in \mathcal{L}_i$ and $L \in \mathcal{L}_j$. We can find $\tilde{L} \in \mathcal{L}(L)$ such that $L' \subset \tilde{L}$, then $I(L) \cap \tilde{L} = \emptyset$ by (4.16). Therefore $I(L) \cap L' = \emptyset$ as desired.

Conclusion (3) follows trivially from the definition of the measures μ_j and μ_{j+1} .

Next we are proving (4). Let $I \in \Xi_L^+$ and let $L'(I)$ be as in (4.15), then

$$\mu_{j+1}(L'(I)) = \mu_{L'(I)}^{6^{j+1}}(L'(I)) = 6^{j+1} \frac{2}{3} |L'(I)| = 6^{j+1} \frac{2}{3} \frac{1}{4} |I| = 6^j |I| = \mu_L^{6^j}(I) = \mu_j(I),$$

where we have used the definition of μ_j and the fact that $\text{jp}(L) \notin I$. We want to make one more comment, estimate (4) is the heart of the measure preserving property.

Proof of (5). Let $L \in \mathcal{L}_j$, then $I(L) \cap L' = \emptyset$ for all $L' \neq L$ by property (2). The following estimates conclude the proof,

$$\begin{aligned} \mu_j(I(L)) &= \mu_L^{6^j}(I(L)) = 6^j \frac{2}{3} |I(L)| \\ \mu_i(I(L)) &= \mu_{I(L)}^{6^j}(I(L)) = 6^j \frac{2}{3} |I(L)|. \end{aligned}$$

The proof of (6) can be deduced from the following equality and a backwards induction argument,

$$(6.2) \quad \mu_{j+1}(L) = \mu_j(L) \quad \text{for all } j \geq 0 \text{ and for all } L \in \mathcal{L}_j.$$

We complete the proof of (6.2) using (4) and (4.7). Let $L \in \mathcal{L}_j$, then $I(\tilde{L}) \cap L = \emptyset$ for all $\tilde{L} \in \mathcal{L}_i$, $i < j$ by property (2) and we get the following

$$\begin{aligned} \mu_{j+1}(L) &= \mu_{I(L)}^{6^j}(L) + \sum_{L' \in \mathcal{L}_{j+1}, L' \subset L} \mu_{L'}^{6^{j+1}}(L) \\ &= \mu_{I(L)}^{6^j}(L) + \sum_{L' \in \mathcal{L}_{j+1}, L' \subset L} \mu_{L'}^{6^{j+1}}(L') \\ &= \mu_{I(L)}^{6^j}(L) + \sum_{I \in \Xi_L^+} \mu_L^{6^j}(I) \\ &= \mu_L^{6^j}(L) = \mu_j(L). \end{aligned}$$

We now turn to proving (7). Let $L \in \mathcal{L}_j$, again a backwards induction argument reduces (7) to prove

$$\mu_{j+1} \left(\left\{ |S_{L,r(L)} \mu_{j+1}| > k \frac{\mu_{j+1}(L)}{|L|} \right\} \right) = \mu_j \left(\left\{ |S_{L,r(L)} \mu_j| > k \frac{\mu_j(L)}{|L|} \right\} \right).$$

The strategy will be to verify $S_{L,r(L)} \mu_{j+1} = S_{L,r(L)} \mu_j$. The rest of the proof follows from (4.12) and properties (5) and (6). This said, we are going to prove

$$(6.3) \quad \langle \mu_{j+1}, h_I \rangle = \langle \mu_j, h_I \rangle \quad \text{for all } I \in \Xi_L.$$

Suppose $I = I(L)$, then $\mu_{j+1}|_{I(L)} = \mu_j|_{I(L)}$ proving (6.3) for this particular case. Suppose $I \in \Xi_L$ but $I \neq I(L)$, then $I^+ \in \Xi_L^+$ and $I^- = \left(I^- \setminus \bigcup_{J \in \Xi_L^+, J \subset I^-} J \right) \cup \bigcup_{J \in \Xi_L^+, J \subset I^-} J$. The decomposition of I^+ and I^- together with property (4) proves the desired estimate

$$\begin{aligned} \langle \mu_{j+1}, h_I \rangle &= \mu_{j+1}(I^+) - \mu_{j+1}(I^-) \\ &= \mu_{j+1}(L'(I^+)) - \mu_{j+1}(I^- \setminus \bigcup_{J \in \Xi_L^+, J \subset I^-} J) - \sum_{J \in \Xi_L^+, J \subset I^-} \mu_{j+1}(J) \\ &= \mu_j(I^+) - \mu_j(I(L)) - \sum_{J \in \Xi_L^+, J \subset I^-} \mu_j(J) \\ &= \mu_j(I^+) - \mu_j(I^-) = \langle \mu_j, h_I \rangle. \end{aligned}$$

□

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